Non-associative skew Laurent polynomial rings

Noncommutative Rings and their Applications VIII, University of Artois, Lens, 2023

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Outline

Background and motivation

Non-commutative rings with a *skewed* or *twisted* multiplication; *Hilbert's twist* [Hil03].

Appear as *universal enveloping algebras* of Lie algebras, *quantized coordinate rings* of affine algebraic varieties, *group rings*, *crossed products* etc. Used e.g. in coding theory.

Ore extensions were introduced by Ore in [Ore33], and *non-associative Ore extensions* in [NÖR18]. What about *non-associative skew Laurent polynomial rings*?

We generalize results on simplicity and Hilbert's basis theorem – with some surprises!

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Convention. All rings in this talk are unital, but not necessarily commutative.

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Skew Laurent polynomial rings

Definition (Skew Laurent polynomial ring)

(S1) S is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$.

(S2) $xR = Rx$.

(S3) S is associative.

Let R be an associative ring with an automorphism σ. The *generalized Laurent polynomial ring* $R[X^{\pm}; \sigma]$ is $\left\{\sum_{i\in\mathbb{Z}} r_i X^i : r_i \in R \text{ zero for all but finitely many } i \in \mathbb{Z}\right\}$. Addition is pointwise and multiplication defined by

$$(rX^m)(sX^n) = (r\sigma^m(s))X^{m+n}, \quad r, s \in R, \ m, n \in \mathbb{Z}.$$

The *generalized polynomial ring* $R[X; \sigma] \subset R[X^{\pm}; \sigma]$ subset of sums with $i \in \mathbb{N}$.

Proposition

$R[X^{\pm}; \sigma]$ is a skew Laurent polynomial ring of R with $x = X$.

Proposition

Every skew Laurent polynomial ring of R is isomorphic to a generalized Laurent polynomial ring $R[X^{\pm}; \sigma]$.

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What are examples of skew Laurent polynomial rings?

Example

Let R be an associative ring. Then $R[X^{\pm}] = R[X^{\pm}; \mathrm{id}_R]$ (and $R[X] = R[X; \mathrm{id}_R]$).

Example

Let $* \colon \mathbb{C} \to \mathbb{C}$, $u \mapsto u^*$ be complex conjugation. In $\mathbb{C}[X^{\pm}; *]$ (and $\mathbb{C}[X; *]$), $Xu = u^*X$.

Example

Let K be a field. The *quantum torus* $T_q(K)$ is $K\langle X^{\pm}, Y^{\pm}\rangle/(XY - qYX)$ for some $q \in K^{\times}$. $T_q(K)$ is (isomorphic to) $K[Y^{\pm}][X^{\pm}; \sigma]$ where σ is the K-automorphism $\sigma(Y) = qY$.

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Non-associative skew Laurent polynomial rings

Convention. A *non-associative ring* is a ring which is not necessarily associative.

Definition (Associator and nuclei)

(A, B, C) finite sums (a, b, c) with $a \in A$, $b \in B$, $c \in C$ for $A, B, C \subseteq R$.

$N_l(R) := \{r \in R \colon (r, s, t) = 0 \text{ for all } s, t \in R\}$. $N_m(R)$ and $N_r(R)$ defined similarly.

Definition (Left R-module)

If R is a non-associative ring, a *left R-module* is an additive group M with a biadditive map $R \times M \to M$, $(r, m) \mapsto rm$ for any $r \in R$ and $m \in M$.

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Definition (Associator and nuclei)

$(\cdot,\cdot,\cdot)\colon R \times R \times R \to R$ is defined by $(r,s,t) := (rs)t - r(st)$ for $r,s,t \in R$.

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Definition (Non-associative skew Laurent polynomial ring)

(N1) S is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$.

(N2) $xR = Rx$.

(N3) $(S, S, x) = (S, x, S) = \{0\}$.

Let R be a **non-associative ring** with an **additive bijection σ that respects 1**.
The *generalized Laurent polynomial ring* $R[X^{\pm}; \sigma]$ is defined as in the associative case,

$$(rX^m)(sX^n) = (r\sigma^m(s))X^{m+n}, \quad r, s \in R, \ m, n \in \mathbb{Z}.$$

The *generalized polynomial ring* $R[X; \sigma] \subset R[X^{\pm}; \sigma]$ subset of sums with $i \in \mathbb{N}$.

Proposition ([BR23])

*$R[X^{\pm}; \sigma]$ is a **non-associative** skew Laurent polynomial ring of R with $x = X$.*

Proposition ([BR23])

*Every **non-associative** skew Laurent polynomial ring of R is isomorphic to a generalized Laurent polynomial ring $R[X^{\pm}; \sigma]$.*

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On $\mathbb{C}$, define $\sigma(a + bi) = a + qbi$ for any $a, b \in \mathbb{R}$ and $q \in \mathbb{R}^{\times}$.
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An *involution* is an anti-automorphism $*\colon R \to R$, $r \mapsto r^*$ s.t. $(r^*)^* = r$ for any $r \in R$. R with an involution $*$ is a $*$*-ring*. Any $*$-ring R gives $R[X^{\pm}; *]$.

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$S:=\mathbb{C}[X^{\pm};\sigma]$ above is not simple if $q=\pm1$ since $S(1+X^4)$ is a non-trivial ideal of S.
S is simple $\iff$ S is not associative.

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(NR1) *R satisfies the ascending chain condition on its right (left) ideals.*

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Can consider *skew power series rings* $R[[X;\sigma]]$ as $\left\{\sum_{i=0}^{\infty} r_i X^i : r_i \in R\right\}$ and *skew Laurent series rings* $R((X;\sigma))$ as $\left\{\sum_{i=k}^{\infty} r_i X^i : r_i \in R, k \in \mathbb{Z}\right\}$ with $(rX^m)(sX^n) = (r\sigma^m(s))X^{m+n}$.

Theorem

Let R be an associative ring with an automorphism σ.
If R is right or left Noetherian, then so are $R[[X;\sigma]]$ and $R((X;\sigma))$.

Theorem ([BR23])

Let R be an associative ring with an additive bijection σ that respects 1.
If R is right Noetherian, then so are $R[[X;\sigma]]$ and $R((X;\sigma))$.

Q: Can one generalize the last theorem for R non-associative?

Q: Can one prove a left version of the last theorem?

Thank you!

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